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1996 J. Phys. A: Math. Gen. 29 6555

(http://iopscience.iop.org/0305-4470/29/20/013)

J. Phys. A: Math. Gen. **29** (1996) 6555–6580. Printed in the UK

The zeta function constructed from the zeros of the Bessel function

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Received 23 April 1996

Abstract. This paper studies the ζ function $\zeta_\alpha(s) = \sum_n (j_{\alpha n}/\pi)^{-s}$ built from the zeros $j_{\alpha n}$ of the Bessel function $J_\alpha(z)$. The known first eight terms of the McMahon expansion $j_{\alpha n} \sim (n+a)[1 - \sum_{p\geqslant 1} b_p(a)(n+a)^{-2p}]$ with $a = (2\alpha - 1)/4$ are used to construct an accurate approximation to $\zeta_\alpha(s)$. The quality of this approximation is investigated numerically by comparison with a known but (at least numerically) little-studied integral formula for $\zeta_\alpha(s)$. Excellent numerical agreement is found for fixed α and variable (real) s, and for fixed s and variable α. Both formulae for $\zeta_\alpha(s)$ therefore seem to work well. Our approximation also accurately reproduces known special values of $\zeta_\alpha(s)$. Important properties of $\zeta_\alpha(s)$ are investigated for the first time, including several of its zeros. In addition, some general theory is presented in two areas: (i) perturbed spectra and (ii) the interrelationship between functions like $J_\alpha(z)$ representable as infinite products, and the ζ functions constructed from their infinite spectrum of zeros.

1. Introduction

Among many possible generalizations of the Riemann zeta function $\zeta(s) = \sum_1^\infty n^{-s}$ to spectra more complicated than the spectrum of positive integers $\{n\}$, one certain to play a major role in physical problems is the 'Bessel ζ function' [1–4] first studied by Hawkins [1]

$$\zeta_\alpha(s) \equiv \sum_{n=1}^{\infty} (j_{\alpha n}/\pi)^{-s} \qquad \mathrm{Re}\, s > 1 \qquad \alpha \geqslant \tfrac{1}{2} \tag{1.1}$$

where $j_{\alpha n}$ are the positive zeros of the Bessel function $J_\alpha(z) : J_\alpha(j_{\alpha n}) = 0$. $\zeta_\alpha(s)$ is sometimes called the Raleigh function, especially for $s = 2, 4, 6, \ldots$ [5–7]. $\zeta_\alpha(s)$ is for $\alpha > \frac{1}{2}$ a continuously, quite nontrivially distorted version of the Riemann ζ function $\zeta(s) = \zeta_{\frac{1}{2}}(s)$. It is clear from $J_{\frac{1}{2}}(z) = (2/\pi z)^{\frac{1}{2}} \sin z$ that $j_{\alpha n}/\pi \to n$, $\zeta_\alpha(s) \to \zeta(s)$ smoothly as $\alpha \to \frac{1}{2}$. For $\alpha > \frac{1}{2}$ the spectrum $\{j_{\alpha n}/\pi\}$ is perturbed away from the spectrum of positive integers $\{n\}$, with the strength of the perturbation increasing as α increases. However, there always remains a one-to-one connection between $\{n\}$ and $\{j_{\alpha n}/\pi\}$. Moreover, the distortion weakens as one looks higher in the spectrum. This is expressed quantitatively by McMahon's asymptotic expansion

$$\begin{aligned} j_{\alpha n}/\pi &\sim (n+a)[1 - \delta_{\alpha n}] \qquad n \gg \alpha \\ \delta_{\alpha n} &= \sum_{p=1}^{\infty} b_p(a)(n+a)^{-2p} \qquad a = \tfrac{1}{4}(2\alpha - 1). \end{aligned} \tag{1.2}$$

0305-4470/96/206555+26$19.50

Standard references [5, 8] usually give the first four coefficients $b_p(a)$. One finds the first seven of them in [9] (see also the appendix below). These functions of a (or α, however we prefer to use parameter a here) are polynomials. The ζ function $\zeta_\alpha(s)$ has its leading or rightmost pole at $s = 1$ with residue $R_0 = 1$, coincident with the pole and residue of $\zeta(s)$. It has additional poles at $s = -1, -3, -5, \ldots$:

$$\zeta_\alpha(1 - 2n + \epsilon) = \frac{1}{\epsilon} R_n(a) + C_n(a) + \epsilon D_n(a) + \cdots \qquad n = 0, 1, 2, \ldots. \tag{1.3}$$

The residues $R_n(a)$ are polynomials in a (sometimes called Hawkins' polynomials [2, 10]) constructed from the b_p. The first few residues have been known for some time [1–4]. All the R_n with $n > 0$ necessarily have a (simple) zero at $a = 0$ or $\alpha = \frac{1}{2}$. At present, little information is available on the finite parts $C_n(a)$ which are far more complicated than polynomials. These, like the R_n, are needed for quantum field theory. Moreover, just as $\zeta(s)$ has the especially simple special values $\zeta(-2n) = 0$, $n = 1, 2, 3, \ldots$, $\zeta_\alpha(s)$ has polynomial values $\zeta_\alpha(-2n)$ at the same points [1, 2]. These (Hawkins) polynomials have simple zeros at $a = 0$ ($\alpha = \frac{1}{2}$). Finally, there are the Raleigh functions [5–7] $\zeta_\alpha(2n) = P_n(\alpha)/Q_n(\alpha)$ which are rational functions of α. The Raleigh polynomials $P_n(\alpha)$ in the numerator have an interesting structure while the denominator $Q_n(\alpha)$ is a relatively uninteresting product of binomials.

The main purpose of the present article is to gain increased computational control over $\zeta_\alpha(s)$ in the region $\mathrm{Re}\, s < 1$ accessible only by analytic continuation of the defining series (1.1). For the reader's convenience and our own we first review in section 2, as compactly as possible, the bulk of existing work on $\zeta_\alpha(s)$. Then we use the asymptotic expansion (1.2) to obtain an approximation to $\zeta_\alpha(s)$ based on the Hurwitz ζ function $\zeta(s, a)$. This approximation makes analytic continuation explicit and straightforward. The use of equation (1.2) in this way is not new. Previous authors have either expanded in powers of n^{-1} rather than $(n + a)^{-1}$ (which is numerically less efficient for large a) or they have used very few terms in equation (1.2) (which of course sharply diminishes accuracy). Our goal is computational capability, for application in quantum field theory. In pursuing it we have taken this use of the McMahon expansion to a new level.

In section 4 we construct from the seven known coefficients $b_p(a)$ in equation (1.2) our approximation to $\zeta_\alpha(s)$. The notation of equation (1.2) is retained so that, when additional $b_p(a)$ for $p \geqslant 8$ are known, one can easily improve the approximation. We shall compare our approximation numerically with an integral formula [11] for $\zeta_\alpha(s)$ reviewed in section 2. The agreement is quite good out to $\alpha = 50$ and higher for the s values tested, and over the range $-6 < s < 1$ for selected α values.

Our work in section 4 makes use of a general procedure which can be applied to an arbitrary distortion $\{\lambda_n\} \to \{\lambda_n(1 - \delta_n)\}$, $\delta_n \to 0$ as $n \to \infty$, of any known spectrum $\{\lambda_n\}$. This part of the derivation is presented separately in an appropriately general notation in section 3. It should be useful in other problems as well.

Let us now turn to physics, and one type of physical problem in which the spectrum (1.2) and other comparably distorted spectra occur.

The spectrum $\{n\}$ of positive integers is the $\{[\text{eigenvalue}]^{\frac{1}{2}}\}$ of the operator $[-\mathrm{d}^2/\mathrm{d}x^2]$ on the interval $[0, \pi]$ with Dirichlet endpoints. The eigenfunctions are $\phi_n(x) \propto \sin(nx)$. Note that this spectral problem can be recast in Schrödinger form

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + D(x) + D(x - \pi) + V(x)\right]\phi_n(x) = k_n^2 \phi_n(x) \qquad -\infty < x < \infty \tag{1.4}$$

with $V(x) = 0$ everywhere and $k_n = n$ in the interval $0 \leqslant x \leqslant \pi$. Outside this interval $k_n \geqslant 0$ is continuous. $D(x)$ and $D(x - \pi)$ represent the Dirichlet conditions at the points $x = 0, \pi$, respectively.

The spectrum $\{j_{\alpha n}/\pi\}$ is the $\{[\text{eigenvalue}]^{\frac{1}{2}}\}$ of the Schrödinger problem (1.4) with Bessel potential $V_{\mathrm{B}}(x) = (\alpha^2 - \frac{1}{4})/x^2$. There are still Dirichlet conditions at $x = 0, \pi$. The condition $D(x)$ is redundant unless $\alpha = \frac{1}{2}$, in which case we are back to the system with two Dirichlet point boundaries. For $\alpha > \frac{1}{2}$ the eigenfunctions are $\phi_n(x) = \text{constant}(x)^{\frac{1}{2}} J_\alpha(k_n x)$ with $k_n = j_{\alpha n}/\pi$ in the interval $0 \leqslant x \leqslant \pi$.

In the general Schrödinger problem (1.4) with impenetrable boundaries at $x = 0, \pi$ the static potential $V(x) \geqslant 0$ represents static spatial structure semitransparent to the wavefunction. If we are doing quantum field theory (hereafter QFT), then the Dirichlet boundaries at $x = 0, \pi$ and the background spatial structure represented by $V(x)$ become parts of a Casimir system in which the quantum field $\hat{\phi}(x)$ constructed from the modes $\phi_n(x)$ interacts with—and is distorted into spatial nonuniformity by—all of these objects. The field $\hat{\phi}(x)$ exerts back forces, or *Casimir forces*, on all objects causing its distortion. The Bessel potential $V_{\mathrm{B}}(x)$ above, which leads to the spectrum $\{j_{\alpha n}/\pi\}$, can be interpreted as 'surface texture' added to the Dirichlet boundary at $x = 0$ by the introduction of $V_{\mathrm{B}}(x)$. We call this 'semihardening' of the Dirichlet boundary [12–14]. The field-theory aspect will be de-emphasized in this article, but it is always present in the authors' thinking, and it perhaps should be in the reader's as well. The spectrum $\{j_{\alpha n}/\pi\}$ is generated by the Bessel potential. A different semihard potential $V(x)$ would generate a different spectrum $\{k_n\}$. There is always a one-to-one connection between the spectrum $\{k_n\}$ and the semihard potential $V(x)$ used. The latter is, by definition, a potential with $V(0) = \infty$ (like $V_{\mathrm{B}}(x)$) and typically $V(x)$ decreases to zero far from the boundary. Ultimately we want to think in terms of arbitrary distortions of spectra which can be understood in the context of the Schrödinger problem (1.4) as expressing some change in the potential $V(x)$. This is the main physical background for the work presented in the present article.

2. Infinite products and their zeta functions

A function such as

$$J_\alpha(z) = \frac{z^\alpha}{2^\alpha \Gamma(\alpha + 1)} \prod_{n=1}^{\infty} \left[1 - \frac{z^2}{j_{\alpha n}^2}\right] \tag{2.1}$$

which has infinitely many zeros and an infinite product representation in terms of them also (under certain assumptions about the spectrum of zeros) has a ζ function—$\zeta_\alpha(s)$ in the case of $J_\alpha(z)$. Although very different from each other $J_\alpha(z)$ and $\zeta_\alpha(s)$ are deeply interrelated and either function can be used to study the other. The integral formula (2.31) below shows particularly clearly how to obtain $\zeta_\alpha(s)$ from $J_\alpha(z)$. Equations (2.22) and (2.25) below show how $J_\alpha(z)$ is computed from $\zeta_\alpha(s)$. Certainly the interrelationship between $\zeta_\alpha(s)$ and $J_\alpha(z)$ is not specific to these functions, but rather is common to all functions which have infinite product representations and ζ functions built from their spectrum of zeros. For this reason we begin our discussion at a fairly general level, and later specialize to the Bessel function.

2.1. General theory

Consider an entire function $F(z)$ with infinitely many zeros on the positive real axis at $z = a_n > 0$ and infinitely many zeros on the negative real axis at $z = -b_n < 0$:

$$F(z) = \left\{\prod_n \left[1 - \frac{z}{a_n}\right] \mathrm{e}^{z/a_n}\right\} \left\{\prod_m \left[1 + \frac{z}{b_m}\right] \mathrm{e}^{-z/b_m}\right\} \qquad 0 < a_1 \leqslant a_2 \leqslant a_3 \leqslant \ldots \qquad 0 < b_1 \leqslant b_2 \leqslant b_3 \leqslant \ldots. \tag{2.2}$$

The zeros $z = a_n$ may be simple or they may be of higher order. In the latter case two or more of the a_n coincide. The same is true of the zeros b_m. For simplicity $F(0) = 1$ and $F'(0) = 0$ are chosen. This representation—a standard topic in analysis (see e.g. [15])—assumes the spectra $\{a_n\}$, $\{b_m\}$ of zeros extend all the way to infinity. This makes these spectra suitable for defining ζ functions:

$$\begin{aligned} A(s) &\equiv \sum_n a_n^{-s} \qquad \mathrm{Re}\, s > 1 \\ B(s) &\equiv \sum_m b_m^{-s} \qquad \mathrm{Re}\, s > 1. \end{aligned} \tag{2.3}$$

Although one could be more general, we choose to work within the framework of boundary semihardening which assumes an asymptotic behaviour like that in equation (1.2):

$$\begin{aligned} a_n &\sim n\pi[1 + \delta_n] \qquad \delta_n \to 0 \qquad n \to \infty \\ b_m &\sim m\pi[1 + \delta'_m] \qquad \delta'_m \to 0 \qquad m \to \infty. \end{aligned} \tag{2.4}$$

Let us also emphasize that both the entire function (2.2) and the ζ functions (2.3) are *completely* determined by the spectra $\{a_n\}$, $\{b_m\}$.

Taking the log of equation (2.2) yields

$$\begin{aligned} \ln F(z) &= \sum_n \left[\frac{z}{a_n} + \ln\left(1 - \frac{z}{a_n}\right)\right] + \sum_m \left[-\frac{z}{b_m} + \ln\left(1 + \frac{z}{b_m}\right)\right] \\ &= -\sum_{k=2}^{\infty} z^k \tfrac{1}{2}[A(k) + (-)^k B(k)]. \end{aligned} \tag{2.5}$$

Here both series $A(k)$, $B(k)$ are convergent for $k \geqslant 2$. Moreover, the expansions of $\ln(1 - z/a)$ and $\ln(1 + z/b)$ converge for $(z/a)^2 < 1$ and $(z/b)^2 < 1$, respectively. Thus for $|z| < \min(a_1, b_1)$ one can freely commute summations, and the result is equation (2.5) which is exact. The coefficients in the ascending power series for $\ln F(z)$ are given by special values $A(k)$, $B(k)$ of the ζ functions (2.3).

If the spectra $\{a_n\}$, $\{b_m\}$ are identical ($a_n = b_n$) as is the case for the Bessel function (2.1), then $A(s) = B(s)$ and equations (2.2), (2.5) become

$$F(z) = \prod_n [1 - z^2/a_n^2] \tag{2.6}$$

$$\ln F(z) = -\sum_{k=1}^{\infty} \frac{1}{k} z^{2k} A(2k). \tag{2.7}$$

Keeping $a_n = b_n$ let us now do something less conservative mathematically. First define

$$G(z) = F(\mathrm{i}z) = \prod_n [1 + z^2/a_n^2] \tag{2.8}$$

whose logarithm is

$$\ln G(z) = \sum_{k=1}^{\infty} (-)^{k+1} \frac{1}{k} z^{2k} A(2k). \tag{2.9}$$

This ascending power series can, of course, be obtained directly from equation (2.7). More interesting is to find the asymptotic series for $\ln G(z)$ in $1/z$:

$$\begin{aligned} \ln G(z) &= \sum_n \left[\ln \frac{z^2}{a_n^2} + \ln\left(1 + \frac{a_n^2}{z^2}\right)\right] \\ &= A(0) \ln z^2 + 2A'(0) + \lim_{s\to 0} \sum_n a_n^{-s} \ln(1 + a_n^2/z^2) \end{aligned} \tag{2.10}$$

where

$$A(0) = \sum_n (a_n)^0 \qquad A'(0) = -\sum_n \ln a_n. \tag{2.11}$$

To evaluate the remaining mode sum on the right in equation (2.10) in terms of $A(s)$ we recognize this series as belonging to the class first studied for a general spectrum in [16]. In the notation of that paper,

$$f(z) = \ln(1+z^2) = \sum_{k=0}^{\infty} (-)^k \frac{1}{k+1} z^{2k+2}$$

so that $C(k) = (k+1)^{-1}$ and $a = b = 2$. Moreover $B = 1$ follows from equation (2.4). Directly from equations (2.1), (2.3) and (2.5*a*) of [16] we find

$$\lim_{s\to 0} \sum_n a_n^{-s} \ln(1 + a_n^2/z^2) = \sum_{k=0}^{\infty} (-)^k \frac{1}{(k+1)z^{2k+2}} A(-2k-2)$$
$$+ \left\{ -\sum_{n=0}^{\infty} \frac{\pi r_n}{(\Delta_n - 1) z^{\Delta_n - 1} \cos\frac{\pi}{2}\Delta_n} \right\}_p + \{\ \}_{\mathrm{ex}} \tag{2.12}$$

where some labelling from [16] is temporarily retained for clarity. In the curly bracket $\{\}_p$ the r_n are residues of the poles of $A(s)$:

$$A(1 - \Delta_n + \epsilon) = \frac{1}{\epsilon} r_n + c_n + \mathrm{O}(\epsilon) \qquad n = 0, 1, 2, \ldots \qquad 0 = \Delta_0 < \Delta_1 < \Delta_2 < \cdots. \tag{2.13}$$

The asymptotic behaviour assumed in equation (2.4) dictates that the rightmost pole is at $s = 1$ and has residue $r_0 = 1/\pi$. Pole positions $s = 1 - \Delta_n$ and residues further to the left depend on the spectrum $\{a_n\}$. The other curly bracket $\{\}_{\mathrm{ex}}$ in equation (2.12) represents an unknown exponentially small (in z) function which we discard, giving our final result for $\ln G(z)$ the character of an asymptotic series:

$$\ln G(z) \sim A(0)\ln z^2 + 2A'(0) + z + \sum_{k=0}^{\infty} (-)^k \frac{1}{(k+1)z^{2k+2}} A(-2k-2)$$
$$- \sum_{n=1}^{\infty} \frac{\pi r_n}{(\Delta n - 1) z^{\Delta_n - 1} \cos\frac{\pi}{2}\Delta_n}. \tag{2.14}$$

Here for later use we write separately the $n = 0$ contribution ($= z$). Note that this term yields a factor $\exp z$ in $G(z)$ for any spectrum $\{a_n\}$ having asymptotic behaviour (2.4).

Equations (2.9) and (2.14) are rather general results giving $\ln G(z)$ exactly as an ascending power series in z, and asymptotically as a series in $1/z$. The coefficients of both series are expressed in terms of the ζ function $A(s)$. Equations (2.9) and (2.14) display very clearly how $G(z)$ is constructed from $A(s)$. These results can be extended to different spectra $\{a_n\}$, $\{b_m\}$ but we do not wish to do this here. Instead we proceed to the inverse problem of determining $A(s)$ from $G(s)$. This has been discussed in the physics literature in [17] at a general level. More recently, the idea has been applied to $\zeta_\alpha(s)$ [11] and to more complicated ζ functions constructed from $\zeta_\alpha(s)$ [18].

Returning to equation (2.6) we note that

$$\frac{\mathrm{d}}{\mathrm{d}z} \ln F(z) = -2z \sum_n (a_n^2 - z^2)^{-1}$$

$$\mathrm{Re\,s}\left\{ z^{-s} \frac{\mathrm{d}}{\mathrm{d}z} \ln F(z) \right\}_{z=a_n} = a_n^{-s}.$$

Thus the ζ function $A(s)$ can be expressed as a Cauchy integral

$$A(s) = \sum_n a_n^{-s} = \frac{1}{2\pi \mathrm{i}} \oint_H \mathrm{d}z \, \left\{ z^{-s} \frac{\mathrm{d}}{\mathrm{d}z} \ln F(z) \right\} \tag{2.15}$$

where H is a hairpin contour enclosing (counterclockwise) the positive z-axis. Now expand H until it becomes the perimeter $D = L + C$ of the infinite half disk with vertical side L at $\mathrm{Re}\, z = 0$ running from $\mathrm{Im}\, z = \mathrm{i}\infty$ to $-\mathrm{i}\infty$, and infinite semicircle C bounding the right half plane at infinity. We hold $\mathrm{Re}\, s > 0$ and sufficiently large that C does not contribute to the contour integral, leaving the integral representation

$$\begin{aligned} A(s) &= \frac{1}{2\pi \mathrm{i}} \int_{\infty}^{-\infty} \mathrm{d}y (\mathrm{i}y)^{-s} \frac{\mathrm{d}}{\mathrm{d}y} \ln F(\mathrm{i}y) \\ &= \frac{1}{\pi} \sin \frac{\pi s}{2} \int_0^{\infty} \mathrm{d}y \ y^{-s} \ \frac{\mathrm{d}}{\mathrm{d}y} \ln G(y). \end{aligned} \tag{2.16}$$

If one knows the function $G(y)$ well enough, one can evaluate $A(s)$ from it using this formula, without knowing explicitly the spectrum of zeros $\{a_n\}$ of $F(z)$. However, as it stands, the integral (2.16) is not yet defined. We see from equation (2.14) that

$$\frac{\mathrm{d}}{\mathrm{d}y} \ln G(y) \sim \sum_{k=0}^{\infty} (-)^k \frac{2}{y^{2k+1}} A(-2k) + \sum_{n=0}^{\infty} \frac{\pi r_n}{y^{\Delta_n} \cos \frac{\pi}{2} \Delta_n} \tag{2.17}$$

and consequently the integral (2.16) diverges at $y = \infty$ for $\mathrm{Re}\, s > 0$. Fortunately, this problem is easily eliminated by using the identity

$$\int_0^{\infty} \mathrm{d}y \, y^{-s} = 0 \qquad \text{if analytic in } s \tag{2.18}$$

to subtract as many terms as desired from the integrand in equation (2.16). Subtracting all terms with $1/y$ to a power smaller than $2N + 1$ we obtain the following representation for $A(s)$:

$$A(s) = \frac{1}{\pi} \sin \frac{\pi s}{2} \{I_N(s) + P_N(s)\} \tag{2.19}$$

$$I_N(s) \equiv \int_0^{\infty} \mathrm{d}y \, y^{-s} \left\{ \frac{\mathrm{d}}{\mathrm{d}y} \ln G(y) - \Theta(y-1) \left[\sum_{n=0}^{M} \frac{\pi r_n}{y^{\Delta_n} \cos \frac{\pi}{2} \Delta_n} + \sum_{k=0}^{N} (-)^k \frac{2A(-2k)}{y^{2k+1}} \right] \right\}$$
$$-2N < \mathrm{Re}\, s < 2 \tag{2.20}$$

$$P_N(s) \equiv \sum_{n=0}^{M} \frac{\pi r_n}{(s - 1 + \Delta_n) \cos \frac{\pi}{2} \Delta_n} + \sum_{k=0}^{N} (-)^k \frac{2A(-2k)}{s + 2k} \qquad \text{for all } s \tag{2.21}$$

where $\Theta(y) = 1$ (or 0) for $y > 0 (y < 0)$. Here M is such that $\Delta_M \geqslant 2N + 1$. $I_N(s)$ converges by construction at $y = \infty$ for $\mathrm{Re}\, s > -2N$, and at $y = 0$ for $\mathrm{Re}\, s < 2$ because $\ln G(y) \approx y^2 A(2) + \cdots$. $P_N(s)$ is defined throughout the s-plane. Note that in equation (2.19) the poles and residues of $A(s)$ are present in $P_N(s)$ by construction, as are the special values $A(-2k)$. The real virtue of this formula is that it enables one numerically to evaluate $A(s)$ away from its poles and the points $s = -2k$.

2.2. Bessel functions

We now specialize the preceding discussion to the Bessel function problem. Expressing equation (2.1) as $J_\alpha(z) = [z^\alpha / 2^\alpha \Gamma(\alpha + 1)] F(z)$ we find from equation (2.7)

$$\ln J_\alpha(z) - \ln \left[\frac{z^\alpha}{2^\alpha \Gamma(\alpha + 1)} \right] = -\sum_{k=1}^{\infty} \frac{1}{k} \left(\frac{z}{\pi} \right)^{2k} \zeta_\alpha(2k). \tag{2.22}$$

Here $a_n = j_{\alpha n}$ and $A(s) = \pi^{-s}\zeta_\alpha(s)$ in the more general notation. Exponentiating equation (2.22) and comparing with the power series

$$J_\alpha(z) = \left(\frac{z}{2}\right)^\alpha \sum_{k=0}^{\infty} \frac{(-z^2/4)^k}{k!\Gamma(\alpha+k+1)} \tag{2.23}$$

one easily obtains the first few special values $\zeta_\alpha(2n)$, $n = 1, 2, \ldots$; e.g.

$$\begin{aligned}\frac{1}{\pi^2}\zeta_\alpha(2) &= \sum_{n=1}^{\infty} j_{\alpha n}^{-2} = \frac{1}{4(\alpha+1)}\\ \frac{1}{\pi^2}\zeta_\alpha(4) &= \sum_{n=1}^{\infty} j_{\alpha n}^{-4} = \frac{1}{4^2(\alpha+1)^2(\alpha+2)}.\end{aligned} \tag{2.24}$$

Such series (given up to $n = 10$ in [3]) are known as Raleigh series [5]. Conversely, from the special values $\zeta_\alpha(2n)$ and equation (2.22) one can recover the power series (2.23).

Recalling that $I_\alpha(z) = [z^\alpha/2^\alpha\Gamma(\alpha+1)]G(z)$ we find from equation (2.14)

$$\begin{aligned}\ln I_\alpha(z) - \ln[z^\alpha/2^\alpha\Gamma(\alpha+1)] \sim -\ln\left(\frac{z}{\pi}\right)^{\alpha+1/2} + 2\zeta_\alpha'(0) + z\\ + \sum_{k=1}^{\infty}(-)^{k+1}\frac{1}{k}\left(\frac{\pi}{z}\right)^{2k}\zeta_\alpha(-2k) - \sum_{n=1}^{\infty}(-)^n\frac{R_n}{(2n-1)}z\left(\frac{\pi}{z}\right)^{2n}.\end{aligned} \tag{2.25}$$

Here the R_n are the residues of $\zeta_\alpha(s)$ in equation (1.3). Also (to speed things up) we have used the known special value $\zeta_\alpha(0) = -\frac{1}{2}(\alpha + \frac{1}{2})$ in the first term on the right. We now exponentiate equation (2.25) and compare it with the asymptotic expansion for fixed order α:

$$\begin{aligned}I_\alpha(z)\sqrt{2\pi z}\mathrm{e}^{-z} \sim 1 - \frac{1}{2z}(\alpha^2 - 1/4) + \frac{1}{2!(2z)^2}(\alpha^2 - \tfrac{1}{4})(\alpha^2 - \tfrac{9}{4})\\ - \frac{1}{3!(2z)^3}(\alpha^2 - \tfrac{1}{4})(\alpha^2 - \tfrac{9}{4})(\alpha^2 - \tfrac{25}{4}) + \cdots.\end{aligned} \tag{2.26}$$

From the $z^0 = 1$ term one finds the special value

$$\zeta_\alpha'(0) = \frac{1}{2}\ln\left[\frac{2^\alpha\Gamma(\alpha+1)}{\sqrt{2}\pi^{\alpha+1}}\right] \tag{2.27}$$

which evidently was not previously known. The $\alpha \to \frac{1}{2}$ limit of equation (2.27) is $\zeta'(0) = -\frac{1}{2}\ln(2\pi)$ as it should be. A comparison of $1/z$ terms yields the residue

$$R_1 = -\frac{1}{2\pi^2}(\alpha^2 - \tfrac{1}{4}). \tag{2.28}$$

From the $1/z^2$ terms we find

$$\zeta_\alpha(-2) = -\frac{1}{4\pi^2}(\alpha^2 - \tfrac{1}{4}). \tag{2.29}$$

Comparing $1/z^3$ terms yields

$$R_2 = -\frac{1}{8\pi^4}(\alpha^2 - \tfrac{1}{4})(\alpha^2 - \tfrac{25}{4}). \tag{2.30}$$

In this way one can systematically extract the pole residues R_n and special values $\zeta_\alpha(-2k)$ from the asymptotic series (2.26) for $I_\alpha(z)$, which is known to all orders. There are, of course, other (closely related) ways to obtain these same quantities [1–4, 11].

In the appendix we list the residues R_n of $\zeta_\alpha(s)$ for $0 \leqslant n \leqslant 7$ and the special values $\zeta_\alpha(-2k)$ for $0 \leqslant k \leqslant 6$. Regarding R_n and $\zeta_\alpha(-2k)$ as known functions, and remembering that $A(s) = \pi^{-s}\zeta_\alpha(s)$, $r_n = \pi^{2n-1}R_n$, $\Delta_n = 2n$ and the connection between $G(y)$ and $I_\alpha(y)$ we rewrite equation (2.19) as an integral representation for $\zeta_\alpha(s)$:

$$\zeta_\alpha(s) = \pi^{s-1}\sin\frac{\pi s}{2}\{I_N(s) + P_N(s)\} \tag{2.31}$$

$$\begin{aligned} I_N(s) = \int_0^\infty \mathrm{d}y\, y^{-s}\Bigg\{\frac{\mathrm{d}}{\mathrm{d}y}\ln\left[\frac{2^\alpha}{y^\alpha}\Gamma(\alpha+1)I_\alpha(y)\right] \\ -\Theta(y-1)\left[\sum_{n=0}^{N+1}(-)^n\frac{\pi^{2n}R_n}{y^{2n}} + \sum_{k=0}^{N}(-)^k\frac{2\pi^{2k}\zeta_\alpha(-2k)}{y^{2k+1}}\right]\Bigg\} \\ -2N < \mathrm{Re}\, s < 2 \end{aligned} \tag{2.32}$$

$$P_N(s) = \sum_{n=0}^{N+1}(-)^n\frac{\pi^{2n}R_n}{s-1+2n} + \sum_{k=0}^{N}(-)^k\frac{2\pi^{2k}\zeta_\alpha(-2k)}{s+2k}. \tag{2.33}$$

An equivalent integral representation was given in [11] (see also [18]). Equations (2.31)–(2.33) will be tested numerically in section 5 for N-dependence (which appears to be weak), and will be compared with the very different representation of $\zeta_\alpha(s)$ derived in section 4 and seen to agree well wherever tested.

To check various things above we set $\alpha = \frac{1}{2}$. Equations (2.1) and (2.22) then become the known formulae

$$\sin z = z\prod_{n=1}^{\infty}[1 - z^2/n^2\pi^2] \qquad \ln\left[\sqrt{\frac{\pi}{2z}}J_{1/2}(z)\right] = \ln\left[\frac{\sin z}{z}\right] = -\sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{z}{\pi}\right)^{2k}\zeta(2k).$$

Also, equation (2.25) becomes

$$\ln\left[\sqrt{\frac{\pi}{2z}}I_{1/2}(z)\right] = \ln\left[\frac{\sinh z}{z}\right] = -\ln 2z + z + \ln(1 - \mathrm{e}^{-2z}) \sim -\ln 2z + z$$

which agrees with the right-hand side of equation (2.25). Finally, equation (2.31) reads

$$\zeta(s) = \pi^{s-1}\sin\frac{\pi s}{2}\{I_0(s) + P_0(s)\} \tag{2.34}$$

where

$$\begin{aligned} I_0(s) &= \int_0^\infty \mathrm{d}y\, y^{-s}\left\{\frac{\mathrm{d}}{\mathrm{d}y}\ln\left[\frac{1}{y}\sinh y\right] - \Theta(y-1)[1 - 1/y]\right\} \\ &= \int_0^\infty \mathrm{d}y\, y^{-s}\left\{\frac{2}{\mathrm{e}^{2y}-1} + [1 - \Theta(y-1)][1 - 1/y]\right\} \qquad 0 < \mathrm{Re}\, s < 2 \end{aligned} \tag{2.35}$$

and

$$P_0(s) = \frac{1}{s-1} - \frac{1}{s} = -\int_0^1 \mathrm{d}y\, y^{-s}\left[1 - \frac{1}{y}\right]. \tag{2.36}$$

Formally

$$I_0(s) + P_0(s) = \int_0^\infty \mathrm{d}y\, y^{-s}\frac{2}{\mathrm{e}^{2y}-1} = 2^s\Gamma(1-s)\zeta(1-s)$$

so that equation (2.34) becomes the reflection formula for $\zeta(s)$. However, equation (2.34) with $I_0(s)$ and $P_0(s)$ as in equations (2.35), (2.36) is an explicit representation for $\zeta(s)$ in the strip $0 < \mathrm{Re}\, s < 2$ without analytic continuation.

3. Perturbed spectra

Consider an arbitrary spectrum $\{\lambda_m\}$ whose properties endow it with a ζ function

$$Z(s) \equiv \sum_m \lambda_m^{-s} \qquad \mathrm{Re}\, s > B > 0. \tag{3.1}$$

According to general theory $Z(s)$ has poles only on the real axis, with the rightmost pole at $s = B > 0$ and possible additional poles $s = B - \Delta_n$ with spacings $0 = \Delta_0 < \Delta_1 < \Delta_2 < \cdots$:

$$Z(B - \Delta_n + \epsilon) = \frac{1}{\epsilon} R_n + C_n + \epsilon D_n + \mathrm{O}(\epsilon^2) \qquad \epsilon \to 0 \qquad n = 0, 1, 2, \ldots. \tag{3.2}$$

The point $s = 0$ is never a pole. If the λ_m are eigenvalues of an operator of order d on a manifold of dimension N, then $B = N/d$ and $\Delta_d = n/d$.

Let us define a 'perturbed' version of the original spectrum to be any continuous distortion of the individual eigenvalues

$$\lambda_m \to \lambda_m(1 - \delta_m) \qquad |\delta_m| < 1 \qquad \delta_m \to 0 \qquad \text{as } m \to \infty$$

which vanishes towards the upper end of the spectrum. In the QFT language mentioned in the introduction, perturbing a spectrum may correspond to semihardening a hard boundary, or altering the semihardening of an already semihard boundary. The effect on the ζ function (3.1) is to replace it by another one

$$Z_\delta(s) \equiv \sum_m \lambda_m^{-s}(1 - \delta_m)^{-s} \qquad \mathrm{Re}\, s > B \tag{3.3}$$

whose pole structure

$$Z_\delta(B - \Delta_{\delta n} + \epsilon) = \frac{1}{\epsilon} R_{\delta n} + C_{\delta n} + \epsilon D_{\delta n} + \mathrm{O}(\epsilon^2) \tag{3.4}$$

and other properties may differ greatly from those of $Z(s)$. However, the abscissa of convergence $\mathrm{Re}\, s = B$ or the position $s = B$ of the rightmost pole and the residue $R_{\delta 0} = R_0$ of this pole are certain to remain unaffected by the perturbation of the spectrum. This is because the original and perturbed spectra are asymptotically the same. When the perturbation of the original spectrum is removed (all $\delta_m \to 0$)$Z_\delta(s)$ smoothly becomes $Z(s)$.

It is helpful to rewrite equation (3.3) in the form

$$\begin{aligned} Z_\delta(s) - Z(s) &= \sum_{k=1}^{\infty} \frac{\Gamma(s+k)}{k!\Gamma(s)} f_k(s) \\ &= s f_1(s) + \frac{1}{2} s(s+1) f_2(s) + \frac{1}{3!} s(s+1)(s+2) f_3(s) + \cdots \end{aligned} \tag{3.5}$$

where each term is a simple polynomial in s multiplying a function

$$f_k(s) \equiv \sum_m \lambda_m^{-s} (\delta_m)^k \qquad k \geqslant 1 \qquad \text{for all } s \text{ after analytic continuation} \tag{3.6}$$

which remains to be calculated. Here we have used the (convergent) binomial expansion $\sum_k$ of $(1 - \delta_m)^{-s}$ followed by commutation of the (for $\mathrm{Re}\, s > B$ convergent) mode sum $\sum_m$ through $\sum_k$ which is allowable for $\mathrm{Re}\, s > B$. Once equation (3.5) has been reached the continuation to $\mathrm{Re}\, s < B$ is explicit as long as the functions $f_k(s)$ can be continued (and we henceforth assume this is possible). We also assume the $f_k(s)$ are meromorphic functions having only simple poles, and illustrate this later with our $\zeta_\alpha(s)$ example.

Equation (3.5)—the central formula of this section—is not a deep result but it is definitely useful, especially in the analytic continuation region $\mathrm{Re}\, s < B$ where one typically needs help in calculating ζ functions. The implementation of equation (3.5) of course requires that one compute the functions $f_k(s)$ in equation (3.6). Without becoming more specific about the spectrum $\{\lambda_m\}$ and its distortion $\{\delta_m\}$ one cannot really calculate further. However, there are useful things to be said even at this general level.

Equation (3.5) is an expansion in the perturbation $\{\delta_m\}$ in its entirety. $f_1(s)$ is of order $\{\delta_m\}$, $f_2(s)$ is of order $\{\delta_m^2\}$, and so on. As $\delta_m \to 0$ (for all m) the $f_k(s)$ all vanish. Thus for a small perturbation it may be possible to approximate $Z_\delta(s)$ rather effectively by retaining only a few terms in equation (3.5). One would do this in $\mathrm{Re}\, s < B$, not in $\mathrm{Re}\, s > B$.

An optimal situation arises when, as in the Bessel problem (1.2), the perturbation can be expressed in terms of the unperturbed spectrum

$$\delta_m = \sum_{p\geqslant 1} b_p \lambda_m^{-cp} \qquad c > 0. \tag{3.7}$$

Here the coefficients b_p determine the perturbation. The $f_k(s)$ can in such cases be expressed in terms of the unperturbed ζ function:

$$\begin{aligned} f_k(s) &= \sum_{p_i\geqslant 1} b_{p_1} \cdots b_{p_k} Z(s + cp_1 + \cdots + cp_k) \\ &= [b_1]^k Z(s + ck) + k[b_1]^{k-1} b_2 Z(s + c(k+1)) + \cdots . \end{aligned} \tag{3.8}$$

A good example of the utility of equation (3.5) is the special value $Z_\delta(0)$. From equation (3.5) and the assumed meromorphic nature of $f_k(s)$ we easily find

$$\begin{aligned} Z_\delta(\epsilon) - Z(\epsilon) = {} & [r_1 + \tfrac{1}{2}r_2 + \tfrac{1}{3}r_3 + \tfrac{1}{4}r_4 + \cdots]_{q=0} \\ & + \epsilon[(\tfrac{1}{2}r_2 + \tfrac{1}{2}r_3 + \tfrac{11}{24}r_4 + \cdots) + (c_1 + \tfrac{1}{2}c_2 + \tfrac{1}{3}c_3 + \tfrac{1}{4}c_4 + \cdots)]_{q=0} \\ & + \epsilon^2[(\tfrac{1}{6}r_3 + \tfrac{1}{4}r_4 + \cdots) + (\tfrac{1}{2}c_2 + \tfrac{1}{2}c_3 + \tfrac{11}{24}c_4 + \cdots) \\ & + (d_1 + \tfrac{1}{2}d_2 + \tfrac{1}{3}d_3 + \tfrac{1}{4}d_4 + \cdots)]_{q=0} + \mathrm{O}(\epsilon^3) \end{aligned} \tag{3.9}$$

where r_k, c_k, d_k and e_k are the functions defined (for $q = 0$) by

$$f_k(q + \epsilon) = \frac{1}{\epsilon} r_k(q) + c_k(q) + \epsilon d_k(q) + \epsilon^2 e_k(q) + \mathrm{O}(\epsilon^3). \tag{3.10}$$

Here the notation allows but does not require $f_k(s)$ to have a pole at $s = 0$. If none of these functions has poles at $s = 0$ then $Z_\delta(0) = Z(0)$; the spectral distortion does not affect this special value. This happens to be the case for the Bessel ζ function. More generally, for spectra with the asymptotic behaviour (2.4) whose ζ functions have their rightmost poles at $s = 1$, equation (3.8) shows that any perturbation (3.7) with $c > 1$ will leave the special value $Z_\delta(0) = Z(0)$ unchanged. However, this never happens for the derivative

$$Z'_\delta(0) - Z'(0) = [\tfrac{1}{2}r_2 + \tfrac{1}{2}r_3 + \tfrac{11}{24}r_4 + \cdots]_{q=0} + [c_1 + \tfrac{1}{2}c_2 + \tfrac{1}{3}c_3 + \tfrac{1}{4}c_4 + \cdots]_{q=0}. \tag{3.11}$$

Equation (3.9) yields a corresponding expression for $Z''_\delta(0) - Z''(0)$. Higher derivatives are also computable. Certainly one is not limited to the accuracy displayed in equation (3.9). If the $f_k(s)$ can be calculated, so can $Z_\delta(s)$ to similar accuracy.

The same kind of analysis can be performed around any point s. Of particular interest for the Bessel ζ function are the points $s = 1, -1, -2$ and -3. We write out the equivalents of equation (3.9) for these points:

$$\begin{aligned}
Z_\delta(1+\epsilon) - Z(1+\epsilon) = {} & \frac{1}{\epsilon}[r_1 + r_2 + r_3 + r_4 + \cdots]_{q=1} \\
& + [(r_1 + \tfrac{3}{2}r_2 + \tfrac{11}{6}r_3 + \tfrac{50}{24}r_4 + \cdots) + (c_1 + c_2 + c_3 + c_4 + \cdots)]_{q=1} \\
& + \epsilon[(\tfrac{1}{2}r_2 + r_3 + \tfrac{35}{24}r_4 + \cdots) + (c_1 + \tfrac{3}{2}c_2 + \tfrac{11}{6}c_3 + \tfrac{50}{24}c_4 + \cdots) \\
& + (d_1 + d_2 + d_3 + d_4 + \cdots)]_{q=1} \\
& + \epsilon^2[(\tfrac{1}{6}r_3 + \tfrac{10}{24}r_4 + \cdots) + (\tfrac{1}{2}c_2 + c_3 + \tfrac{35}{24}c_4 + \cdots) \\
& + (d_1 + \tfrac{3}{2}d_2 + \tfrac{11}{6}d_3 + \tfrac{50}{24}d_4 + \cdots) + (e_1 + e_2 + e_3 + e_4 + \cdots)]_{q=1} \\
& + \mathrm{O}(\epsilon^3)
\end{aligned} \tag{3.12}$$

$$\begin{aligned}
Z_\delta(-1+\epsilon) - Z(-1+\epsilon) = {} & \frac{1}{\epsilon}[-r_1]_{q=-1} + [(r_1 - \tfrac{1}{2}r_2 - \tfrac{1}{6}r_3 - \tfrac{1}{12}r_4 + \cdots) - c_1]_{q=-1} \\
& + \epsilon[(\tfrac{1}{2}r_2 - \tfrac{1}{24}r_4 + \cdots) + (c_1 - \tfrac{1}{2}c_2 - \tfrac{1}{6}c_3 - \tfrac{1}{12}c_4 + \cdots) - d_1]_{q=-1} \\
& + \epsilon^2[(\tfrac{1}{6}r_3 + \tfrac{1}{12}r_4 + \cdots) + (\tfrac{1}{2}c_2 - \tfrac{1}{12}c_4 + \cdots) \\
& + (d_1 - \tfrac{1}{2}d_2 - \tfrac{1}{6}d_3 - \tfrac{1}{12}d_4 + \cdots) - e_1]_{q=-1} + \mathrm{O}(\epsilon^3)
\end{aligned} \tag{3.13}$$

$$\begin{aligned}
Z_\delta(-2+\epsilon) - Z(-2+\epsilon) = {} & \frac{1}{\epsilon}[-2r_1 + r_2]_{q=-2} \\
& + [(r_1 - \tfrac{3}{2}r_2 + \tfrac{1}{3}r_3 + \tfrac{1}{12}r_4 + \cdots) - 2c_1 + c_2]_{q=-2} \\
& + \epsilon[(\tfrac{1}{2}r_2 - \tfrac{1}{2}r_3 - \tfrac{1}{24}r_4 + \cdots) \\
& + (c_1 - \tfrac{3}{2}c_2 + \tfrac{1}{3}c_3 + \tfrac{1}{12}c_4 + \cdots) - 2d_1 + d_2]_{q=-2} \\
& + \epsilon^2[(\tfrac{1}{6}r_3 - \tfrac{1}{12}r_4 + \cdots) + (\tfrac{1}{2}c_2 - \tfrac{1}{2}c_3 - \tfrac{1}{24}c_4 + \cdots) \\
& + (d_1 - \tfrac{3}{2}d_2 + \tfrac{1}{3}d_3 + \tfrac{1}{12}d_4 + \cdots) - 2e_1 + e_2]_{q=-2} + \mathrm{O}(\epsilon^3)
\end{aligned} \tag{3.14}$$

$$\begin{aligned}
Z_\delta(-3+\epsilon) - Z(-3+\epsilon) = {} & \frac{1}{\epsilon}[-3r_1 + 3r_2 - r_3]_{q=-3} \\
& + [(r_1 - \tfrac{5}{2}r_2 + \tfrac{11}{6}r_3 - \tfrac{1}{4}r_4 + \cdots) - 3c_1 + 3c_2 - c_3]_{q=-3} \\
& + \epsilon[(\tfrac{1}{2}r_2 - r_3 + \tfrac{11}{24}r_4 + \cdots) + (c_1 - \tfrac{5}{2}c_2 + \tfrac{11}{6}c_3 - \tfrac{1}{4}c_4 + \cdots) \\
& - 3d_1 + 3d_2 - d_3]_{q=-3} \\
& + \epsilon^2[(r_3 - \tfrac{1}{4}r_4 + \cdots) + (\tfrac{1}{2}c_2 - c_3 + \tfrac{11}{24}c_4 + \cdots) \\
& + (d_1 - \tfrac{5}{2}d_2 + \tfrac{11}{6}d_3 - \tfrac{1}{4}d_4 + \cdots) - 3e_1 + 3e_2 - e_3]_{q=-3} + \mathrm{O}(\epsilon^3).
\end{aligned} \tag{3.15}$$

In each case $r_k = r_k(q), \ldots, e_k = e_k(q)$ for the appropriate q value in equation (3.8). Obviously the sequence of formulae (3.12)–(3.15) can be extended to $s = -4, -5, \ldots$. The utility of these somewhat inelegant looking expressions will be apparent in the next section.

4. An approximation to the Bessel zeta function

Let us rewrite the McMahon expansion (1.2) as an exact formula

$$\begin{aligned}
& j_{\alpha n}/\pi = (n+a)[1 - \delta_{\alpha n}] + e_{\alpha n} \\
& \delta_{\alpha n} = \sum_{p=1}^{\infty} b_p(a)(n+a)^{-2p} \qquad a = \tfrac{1}{4}(2\alpha - 1)
\end{aligned} \tag{4.1}$$

where $e_{\alpha n}$ is an unknown function of α and n which vanishes as $n \to \infty$ faster than any power of $1/n$. Equation (4.1) *defines* $e_{\alpha n}$ as the difference between $j_{\alpha n}$ and its asymptotic series (1.2). This makes $e_{\alpha n}$ computable numerically. These numbers happen to be very small even for small values of n (with the possible exception of large-order α). In later numerical calculations we shall consistently ignore the $e_{\alpha n}$. Nonetheless we carry them along formally in the notation, to be reminded that these terms *are* present and *could* be evaluated, and for use in future work. All poles and residues of $\zeta_\alpha(s)$ are totally independent of $e_{\alpha n}$.

The form of equation (4.1) suggests that one view the Bessel ζ function $\zeta_\alpha(s) = Z_\delta(s)$ as a distortion of the Hurwitz ζ function. In the notation of section 3 we choose $\lambda_n = n+a$ so that the undistorted ζ function

$$Z(s) = \sum_{n=1}^{\infty}(n+a)^{-s} = \zeta(s, a+1) \tag{4.2}$$

is the Hurwitz ζ function. Note the important properties of $\zeta(s, a+1)$ (see e.g. [19]):

$$\begin{aligned}
\zeta(s, a+1) &= \sum_{k=0}^{\infty}(-)^k \frac{\Gamma(s+k)}{k!\Gamma(s)} a^k \zeta(s+k) \\
&= \zeta(s) - sa\zeta(s+1) + \tfrac{1}{2}s(s+1)a^2\zeta(s+2) + \cdots \\
\zeta(1+\epsilon, a+1) &= \frac{1}{\epsilon} - \Psi(a+1) + \epsilon\zeta^{(1)}(1, a+1) + \cdots \\
\zeta(-n, a+1) &= -(n+1)^{-1}B_{n+1}(a+1) \qquad n = 0, 1, 2, \ldots
\end{aligned} \tag{4.3}$$

where $\Psi(a+1) = \Gamma'(a+1)/\Gamma(a+1)$ and the $B_n(a+1)$ are Bernoulli polynomials. In particular

$$\begin{aligned}
&\zeta(0, a+1) = -[a+\tfrac{1}{2}] \qquad \zeta'(0, a+1) = \ln\left[\frac{\Gamma(a+1)}{\sqrt{2\pi}}\right] \\
&\zeta(-1, a+1) = -\tfrac{1}{2}[\tfrac{1}{6} + a(a+1)] \\
&\zeta(-2, a+1) = -\tfrac{1}{3}a(a+\tfrac{1}{2})(a+1) \\
&\zeta(-3, a+1) = \tfrac{1}{4}[\tfrac{1}{30} - a^2(a+1)^2].
\end{aligned} \tag{4.4}$$

Now equation (3.5) becomes

$$\zeta_\alpha(s) - \zeta(s, a+1) = \sum_{k=1}^{\infty}\frac{\Gamma(s+k)}{k!\Gamma(s)} f_k(s) \tag{4.5}$$

where the functions $f_k(s)$ remain to be calculated. From equations (3.6), (4.1)

$$\begin{aligned}
f_1(s) - \Delta f_1(s) &= \sum_{n=1}^{\infty}(n+a)^{-s}\sum_{p=1}^{\infty}(n+a)^{-2p}b_p(a) \\
&= \sum_{p=1}^{\infty} b_p(a)\zeta(s+2p, a+1)
\end{aligned} \tag{4.6}$$

where

$$\Delta f_1(s) \equiv \sum_{n=1}^{\infty} e_{\alpha n}(n+a)^{-s} \tag{4.7}$$

is a small, in principle computable, entire function of s. It is entire because $e_{\alpha n}$ vanishes as $n \to \infty$ faster than any power of n can diverge. Thus the poles of $f_1(s)$ all come from the Hurwitz ζ functions in equation (4.6). The rightmost pole of $f_1(s)$ is at $s = -1$ with residue $r = b_1$, the next pole is at $s = -3$ with residue $r = b_2$, the next at $s = -5$ with residue $r = b_3$ and so on. Continuing,

$$f_2(s) - \Delta f_2(s) = \sum_{p_{1,2}=1}^{\infty} b_{p_1} b_{p_2} \zeta(s + 2p_1 + 2p_2, a + 1) \tag{4.8}$$

where

$$\Delta f_2(s) \equiv \sum_{n=1}^{\infty} e_{\alpha n}(n+a)^{-s} \left\{ 2 \sum_{p=1}^{\infty} b_p (n+a)^{-2p} + e_{\alpha n} \right\} \tag{4.9}$$

is another small computable entire function. Again the poles of $f_2(s)$ can be found by inspection. The rightmost pole is at $s = -3$ with residue $r = b_1^2$, the next pole is at $s = -5$ and has residue $r = 2b_1 b_2$, and so on. Now the general pattern is clear (recall also equation (3.8)):

$$f_k(s) - \Delta f_k(s) = \sum_{p_i=1}^{\infty} b_{p_1} b_{p_2} \cdots b_{p_k} \zeta(s + 2p_1 + 2p_2 + \cdots + 2p_k, a + 1) \tag{4.10}$$

has its rightmost pole at $s = 1 - 2k$ with residue $r = (b_1)^k$, its next pole at $s = -1 - 2k$ with residue $r = kb_2(b_1)^{k-1}$, the next pole at $s = -3 - 2k$, and so on.

A brief word on notation. A Δ in front of any symbol, as in equations (4.8)–(4.10), will always specify the contribution to this quantity from $\{e_{\alpha n}\}$. These small contributions are never evaluated, as mentioned earlier, but with enough effort they could be.

A glance at equations (A.1) in the appendix shows that each $b_p(a)$ has a simple zero at $a = 0$:

$$b_p(a) \equiv a\hat{b}_p(a) = a[\hat{b}_p(0) + a\hat{b}'_p(0) + \tfrac{1}{2}a^2\hat{b}''_p(0) + \cdots] \tag{4.11}$$

where $\hat{b}_p(a)$ is a polynomial in a. This simple zero has to be present because the spectral distortion (4.1) vanishes at $a = 0$ (or $\alpha = \frac{1}{2}$). Consequently, equation (4.10) has the form

$$f_k(s) - \Delta f_k(s) = a^k \hat{f}_k(s) \tag{4.12}$$

where $\hat{f}_k(s)$ is finite and nonzero at $a = 0$. Disregarding $\Delta f_k(s)$ we see that equation (4.5) has the nature of an ascending power series in a, with coefficients $\hat{f}_k(s)$ which also depend on a in some nonpower fashion. One can, of course, expand the $\hat{f}_k(s)$ in powers of a to obtain a pure power series.

Equation (4.5) together with equations (4.6)–(4.12) for the $f_k(s)$ comprise the main result of this section: a formula for $\zeta_\alpha(s)$ which, if one knew all the polynomials $b_p(a)$, should be accurate. Because one does know these polynomials for $p \leqslant 7$, equation (4.5) should provide at least a good approximation to $\zeta_\alpha(s)$. Of course this will have to be tested. In the remainder of this section we examine our approximation to $\zeta_\alpha(s)$ analytically, using it to obtain (i) the first three pole residues and the finite parts of these poles, and (ii) the special values of $\zeta_\alpha(s)$ and $(\mathrm{d}/\mathrm{d}s)\zeta_\alpha(s)$ at $s = 0$ and -2. In the next section we test our approximation numerically.

First let us investigate the residues and finite parts of the poles at $s = 1, -1$ and -3. The notation R_n, C_n and D_n is from equation (1.3).

$s = 1 + \epsilon$. All $r_k(1) = 0$ in equation (3.12) and consequently, given equation (4.3), we find $R_0 = 1$ and

$$C_0(a) + \Psi(a+1) = \sum_{k\geqslant 1} c_k(1) \tag{4.13}$$

$$D_0(a) - \zeta^{(1)}(1, a+1) = [c_1 + \tfrac{3}{2}c_2 + \tfrac{11}{6}c_3 + \tfrac{50}{24}c_4 + \cdots]_{q=1} + \sum_{k\geqslant 1} d_k(1). \tag{4.14}$$

Here $c_k(1) = f_k(1)$ is given by equations (4.6)–(4.10) with $s = 1$, and $d_k = \mathrm{d}f_k/\mathrm{d}s|_{s=1}$. In particular the $k = 1$ terms are

$$\begin{aligned} c_1(1) - \Delta c_1 &= \sum_{p=1}^{\infty} b_p \zeta(1+2p, a+1) \\ d_1(1) - \Delta d_1 &= \sum_{p=1}^{\infty} b_p \zeta'(1+2p, a+1) \end{aligned} \tag{4.15}$$

where the prime always means $\mathrm{d}/\mathrm{d}s$. The $k > 1$ terms are given by equation (4.10) and can obviously be displayed in similar fashion. In our approximation we would keep the terms $1 \leqslant p \leqslant 7$ in equations (4.15) and in the similar formulae for $c_{2,3,\ldots}$ and $d_{2,3,\ldots}$. The functions C_0, D_0 will not be studied further here. However, they have effectively been expressed as power series in a by equations (4.13), (4.14).

$s = -1 + \epsilon$. In equation (3.13) $r_1(-1) = b_1(a)$ and all $r_k(-1) = 0$ for $k > 1$. Thus $R_1(a) = -b_1 = -(\alpha^2 - \frac{1}{4})/2\pi^2$. Also

$$C_1(a) - \zeta(-1, a+1) = b_1(-1) - c_1(-1) \tag{4.16}$$

where

$$c_1(-1) - \Delta c_1 = -b_1\Psi(a+1) + \sum_{p=2}^{\infty} b_p \zeta(2p-1, a+1). \tag{4.17}$$

Moreover,

$$D_1(a) - \zeta'(-1, a+1) = [c_1 - \tfrac{1}{2}c_2 - \tfrac{1}{6}c_3 - \tfrac{1}{12}c_4 + \cdots]_{q=-1} - d_1(-1). \tag{4.18}$$

$s = -3 + \epsilon$. In equation (3.15) $r_1(-3) = b_2$ and $r_2(-3) = b_1^2$ with $r_k(-3) = 0$ for $k \geqslant 3$. The pole at $s = -3$ has residue

$$R_2(a) = -3r_1(-3) + 3r_2(-3) = -\frac{1}{8\pi^4}(\alpha^2 - \tfrac{1}{4})(\alpha^2 - \tfrac{25}{4}).$$

Moreover,

$$C_2(a) - \zeta(-3, a+1) = [r_1 - \tfrac{5}{2}r_2]_{q=-3} + [-3c_1 + 3c_2 - c_3]_{q=-3} \tag{4.19}$$

where

$$\begin{aligned} c_1(-3) + \Delta c_1 &= b_1\zeta(-1, a+1) - b_2\Psi(a+1) + \sum_{p=3}^{\infty} b_p \zeta(-3+2p, a+1) \\ c_2(-3) + \Delta c_2 &= -b_1^2\Psi(a+1) + \sum_{\substack{p_{1,2}=1 \\ \text{exclude}(1,1)}}^{\infty} b_{p_1}b_{p_2}\zeta(-3+2p_1+2p_2, a+1) \\ c_3(-3) + \Delta c_3 &= \sum_{p_i=1}^{\infty} b_{p_1}b_{p_2}b_{p_3}\zeta(-3+2p_1+2p_2+2p_3, a+1). \end{aligned} \tag{4.20}$$

Also

$$\begin{aligned} D_2(a) - \zeta'(-3, a+1) = \tfrac{1}{2}r_2(-3) &+ [c_1 - \tfrac{5}{2}c_2 + \tfrac{11}{6}c_3 - \tfrac{1}{4}c_4 + \cdots]_{q=-3} \\ &+ [-3d_1 + 3d_2 - d_3]_{q=-3}. \end{aligned} \tag{4.21}$$

Clearly this procedure can be extended to the poles at $s = -5, -7, \ldots$ without difficulty (except for the increasing algebraic complexity).

$s = \epsilon$. None of the $f_k(\epsilon)$ has a pole at $\epsilon = 0$, so from equation (3.9) we have $\zeta_\alpha(0) = \zeta(0, a+1) = -(a+\frac{1}{2}) = -(2\alpha+1)/4$. From equation (2.27) and using equation (3.11) and $\zeta'(0, a+1)$ in equation (4.4) one finds

$$\zeta'_\alpha(0) = \frac{1}{2}\ln\left[\frac{\Gamma(4a+2)}{(2\pi)^{2a+1}\Gamma(2a+1)}\right]$$
$$= \ln\left[\frac{\Gamma(a+1)}{\sqrt{2\pi}}\right] + \sum_{k=1}^{\infty} f_k(0) \tag{4.22}$$

a closed-form result against which our approximation can be tested. Combining the logarithms we rewrite equation (4.22) as a known power series

$$\sum_{k=1}^{\infty} f_k(0) = \frac{1}{2}\ln\left[\frac{\Gamma(4a+2)}{(2\pi)^{2a}\Gamma(2a+1)\Gamma^2(a+1)}\right]$$
$$= a[2-\ln 2\pi] + \frac{1}{2}\sum_{n=2}^{\infty}(-)^n a^n \frac{1}{n}\{\zeta(n)[4^n-2^n-2]-4^n\}$$
$$= a[2-\ln 2\pi] + a^2\left[\frac{5\pi^2}{12}-4\right] + \mathrm{O}(a^3). \tag{4.23}$$

Does the left-hand side reproduce every term of this power series? Let us check the first two terms. Because $f_k(0) = 0(a^k)$ we need only consider $f_1(0) + f_2(0)$ on the left. From equations (4.3), (4.6), (4.8) one finds in the notation of equation (4.11)

$$f_1(0) - \Delta f_1(0) = a\sum_{p\geqslant 1}\hat{b}_p(0)\zeta(2p) + a^2\sum_{p\geqslant 1}[-2p\hat{b}_p(0)\zeta(2p+1) + \hat{b}'_p(0)\zeta(2p)] + \mathrm{O}(a^3)$$
$$f_2(0) - \Delta f_2(0) = a^2\sum_{p_{1,2}\geqslant 1}\hat{b}_{p_1}(0)\hat{b}_{p_2}(0)\zeta(2p_1+2p_2) + \mathrm{O}(a^3).$$

In the appendix $\hat{b}_p(0)$ is given for $p \leqslant 7$. One also needs

$$\zeta(2n) = (-)^{n+1}\frac{1}{2}\frac{(2\pi)^{2n}}{(2n)!}B_{2n}$$

where the B_{2n} are Bernoulli numbers. In particular $\zeta(0) = -\frac{1}{2}$, $\zeta(2) = \pi^2/6$, $\zeta(4) = \pi^4/90$, $\zeta(6) = \pi^6/945$, $\zeta(8) = \pi^8/9450$, $\zeta(10) = \pi^{10}/93\,555, \ldots$. Comparison with equation (4.23) yields

$$2-\ln 2\pi = 0.162\,123 \sim \sum_{p\geqslant 1}\hat{b}_p(0)\zeta(2p) = \tfrac{1}{6} - \tfrac{1}{180} + \tfrac{1}{630} - \tfrac{1}{840} + \tfrac{1}{594} - \tfrac{691}{180\,180} + \tfrac{1}{78} + \cdots$$
$$= 0.166\,666, 0.161\,111, 0.162\,698, 0.161\,508, 0.163\,191, 0.159\,356, 0.172\,177$$
$$\text{for } p_{\max} = 1, 2, \ldots, 7 \tag{4.24}$$

$$5\pi^2/12 - 4 = 0.112\,335 \sim \sum_{p\geqslant 1}[-2p\hat{b}_p(0)\zeta(2p+1) + \hat{b}'_p(0)\zeta(2p)]$$
$$+ \sum_{p_{1,2}\geqslant 1}\hat{b}_{p_1}(0)\hat{b}_{p_2}(0)\zeta(2p_1+2p_2)$$
$$= 0.100\,857, 0.112\,322, 0.109\,056, 0.111\,377, 0.108\,201, 0.115\,323, 0.0\,918\,458$$
$$\text{for } p_{\max} = 1, 2, \ldots, 7. \tag{4.25}$$

Evidently nonconvergent, both series reveal a fundamental property of our approximation: it is asymptotic in the cut-off value $p_{\max}$ of the sum over p.

$s = -2 + \epsilon$. In equation (3.14) all the $r_k(-2) = 0$ for $k \geqslant 1$ so that

$$\zeta_\alpha(-2) = -\frac{1}{\pi^2} a(a + \tfrac{1}{2}) = -\tfrac{1}{3} a(a + \tfrac{1}{2})(a + 1) + [-2c_1 + c_2]. \tag{4.26}$$

Here $\zeta(-2, a + 1)$ has been replaced by the polynomial in equation (4.4) and

$$c_1(-2) - \Delta c_1 \equiv \sum_{n=1}^{\infty} a^n h_{1n} = \sum_{p \geqslant 1} b_p(a) \zeta(2p - 2, a + 1) \tag{4.27}$$

$$c_2(-2) - \Delta c_2 \equiv \sum_{n=2}^{\infty} a^n h_{2n} = \sum_{p_{1,2} \geqslant 1} b_{p_1}(a) b_{p_2}(a) \zeta(2p_1 + 2p_2 - 2, a + 1) \tag{4.28}$$

where the hs are independent of a. Equation (4.26) can be checked order by order in a. At O(a)

$$\begin{aligned} \frac{1}{6} - \frac{1}{2\pi^2} = 0.116\,006 \sim -2h_{11} &= -2 \sum_{p \geqslant 1} \hat{b}_p(0) \zeta(2p - 2) \\ &= -\frac{2}{\pi^2} [-\tfrac{1}{2} - \tfrac{1}{12} + \tfrac{1}{60} - \tfrac{1}{84} + \tfrac{1}{60} - \tfrac{5}{132} + \tfrac{691}{5460} + \cdots] \\ &= 0.1013, 0.1182, 0.1148, 0.1172, 0.1139, 0.1215, 0.0959 \\ &\quad \text{for } p_{\max} = 1, 2, \ldots, 7. \end{aligned}$$

At O(a^2)

$$\frac{1}{2} - \frac{1}{\pi^2} = 0.398\,679 \sim -2h_{12} + h_{22}$$

where

$$\begin{aligned} h_{12} &= [\hat{b}'_1(0)\zeta(0) - \hat{b}_1(0)] + \sum_{p \geqslant 2} [\hat{b}'_p(0)\zeta(2p - 2) - (2p - 2)\hat{b}_p(0)\zeta(2p - 1)] \\ &= -0.2026, -0.1875, -0.1944, -0.1871, -0.2007, -0.1621, -0.3164 \\ &\quad \text{for } p_{\max} = 1, 2, \ldots, 7 \\ h_{22} &= \sum_{p_{1,2} \geqslant 1} \hat{b}_{p_1}(0) \hat{b}_{p_2}(0) \zeta(2p_1 + 2p_2 - 2) \\ &= 0.0169, 0.0158, 0.0161, 0.0159, 0.0162, 0.0155, 0.0180 \\ &\quad \text{for } p_{\max} = 1, 2, \ldots, 7. \end{aligned}$$

Again the asymptotic dependence on $p_{\max}$ is evident. From equation (3.14) we find

$$\zeta'_\alpha(-2) - \zeta'(-2, a + 1) = [c_1 - \tfrac{3}{2} c_2 + \tfrac{1}{3} c_3 + \tfrac{1}{12} c_4 + \cdots]_{q=-2} - 2d_1(-2) + d_2(-2)$$

which can be similarly evaluated.

5. Numerical calculations

Sections 2 and 4 present two very different formulae for $\zeta_\alpha(s)$. Both are quite complicated, and it is important to know if these formulae agree numerically. If they do, then one can be rather certain the numerical values obtained for $\zeta_\alpha(s)$ are accurate. We present in this section a sampling of calculations which display the quite good agreement we have found in our work thus far. Excepting [13], $\zeta_\alpha(s)$ has not previously been studied numerically in the literature. Hence these calculations serve the second purpose of providing new insight into and information on $\zeta_\alpha(s)$ itself. We mention that it is possible to vary the approximation (4.5) by varying the $p_{\max}$ used in the sums over p, and to vary the formula (2.31) by changing N. Either variation has visible but usually weak effects: the agreement between equations (2.31) and (4.5) appears to be 'robust'. However, there remain many ranges of s and α untested and it would be premature to say these formulae agree overall. Independently of the integral formula (2.31), we can also test our approximation (4.5) against known special values of $\zeta_\alpha(s)$. This is done for $\zeta_\alpha(0)$, $\zeta'_\alpha(0)$ and $\zeta_\alpha(-2)$ and again equation (4.5) performs well. All calculations based on equation (4.5) are done using $p_{\max} = 7$ unless otherwise specified. All calculations based on equation (2.31) are done using the smallest possible value of N. Throughout this section it will be convenient to use the parameter α (rather than a).

5.1. Zeros of $\zeta_\alpha(s)$

The Bessel ζ function has zeros along the negative s-axis—infinitely many of them, presumably. These have received no attention in the literature. Simple considerations reveal that zeros must exist, and that their enumeration will not be a simple task. The rightmost pole residue $R_0 = 1$ shows that $\zeta_\alpha(s) \to \pm\infty$ as $s \to 1\pm$ for all $\alpha \geqslant \frac{1}{2}$. The residue $R_1 = -(\alpha^2 - \frac{1}{4})/2\pi^2 < 0$ of the pole at $s = -1$ shows that $\zeta_\alpha(s) \to \mp\infty$ as $s \to -1\pm$ for all $\alpha > \frac{1}{2}$. The point $s = 0$ lies midway between $s = 1$ and $s = -1$, and we know that $\zeta_\alpha(0) = -(\alpha + \frac{1}{2})/2 < 0$, becoming more negative as α increases. Evidently there can be no zero between $s = 1$ and $s = -1$. However, between $s = -1$ and $s = -3$ there are one or two zeros, depending on α. The residue R_2 of the pole at $s = -3$ (see equation (A.3)) is $> 0 (= 0, < 0)$ for $\alpha < \frac{5}{2} (= \frac{5}{2}, > \frac{5}{2})$. Therefore

$$\zeta_\alpha(s) \to \begin{matrix} \pm\infty & \frac{1}{2} < \alpha < \frac{5}{2} \\ \mp\infty & \alpha > \frac{5}{2} \end{matrix} \qquad s \to -3\pm .$$

Moreover, we know that $4\pi^2\zeta_\alpha(-2) = -(\alpha^2 - \frac{1}{4}) < 0$ for $\alpha > \frac{1}{2}$. Thus one can easily draw a crude sketch of $\zeta_\alpha(s)$ between $s = 1$ and $s = -3$ without any calculation. $\zeta_\alpha(s)$ has no zeros between the poles $s = 1$ and $s = -1$ for any $\alpha > \frac{1}{2}$. For $\frac{5}{2} > \alpha > \frac{1}{2}$ there must be two zeros between the poles $s = -3$ and $s = -1$. For $\alpha > \frac{5}{2}$ there is only one zero in this interval. For $\alpha = \frac{5}{2}$ the pole at $s = -3$ disappears, but there continues to be one zero between $s = -1$ and $s = -3$.

Extending this sketch further to the left rapidly becomes more difficult. One has enough information (the pole residues and special values $\zeta_\alpha(-2n)$). However, the number of possibilities becomes increasingly unwieldy. We therefore proceed to numerical analysis.

Figures 1–3 show the results of numerical evaluation of $\zeta_\alpha(s)$ for $\alpha = 1$, $\frac{5}{2}$ and 3, respectively, over the interval $-6 < s < 0$ using both representations of $\zeta_\alpha(s)$. The features described above are clearly displayed. One also sees how the slope $\zeta'_\alpha(s)$ depends on α at, for example, the point $s = -2$ which will be discussed below.

5.2. Finite parts of the poles at $s = 1, -1$ and -3

Here we are concerned with the finite parts $C_{0,1,2}(a)$ in equation (1.3) of the first three poles of $\zeta_\alpha(s)$. We have already calculated $C_{0,1,2}(a)$ within our approximation, the results being equations (4.13), (4.16) and (4.19). The corresponding formulae obtained from the integral representation (2.31) are

$$C_n(a) - R_n(a)\ln\pi = (-)^n\pi^{-2n}[I_N(1-2n) + \tilde{P}_N(1-2n)] \tag{5.1}$$

where $\tilde{P}_N(1-2n)$ is given by equation (2.33) with the term containing R_n deleted. In figures 4 and 5 we plot $C_0(a)$ versus a for $\frac{1}{2} < \alpha < 3$ and $\alpha < 50$, respectively, calculated

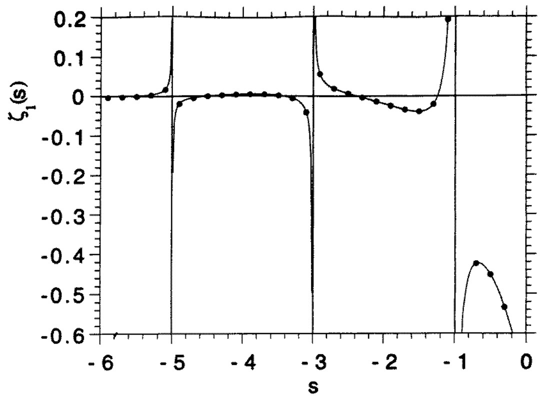

Figure 1. Points: equation (4.5). Line: equation (2.32).

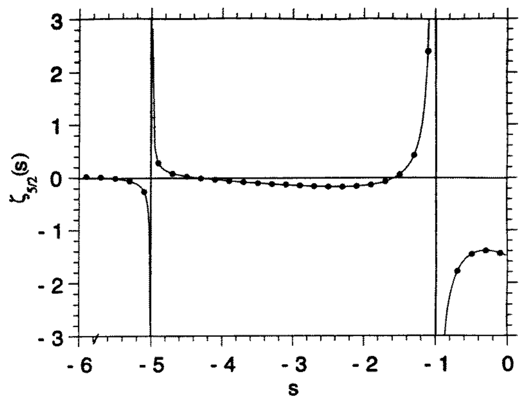

Figure 2. Points: equation (4.5). Line: equation (2.32).

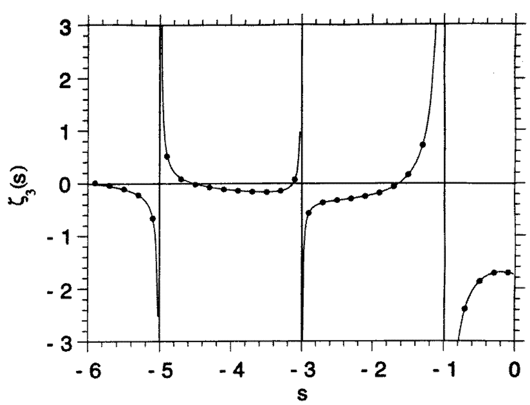

Figure 3. Points: equation (4.5). Line: equation (2.32).

using both representations. The agreement is noteworthy. Figures 6, 7 and 8, 9 do the same thing for $C_1(a)$ and $C_2(a)$. Again the agreement is excellent.

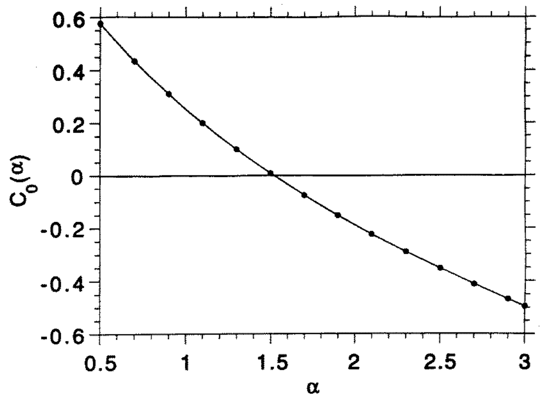

Figure 4. Points: equation (4.13). Line: equation (5.1).

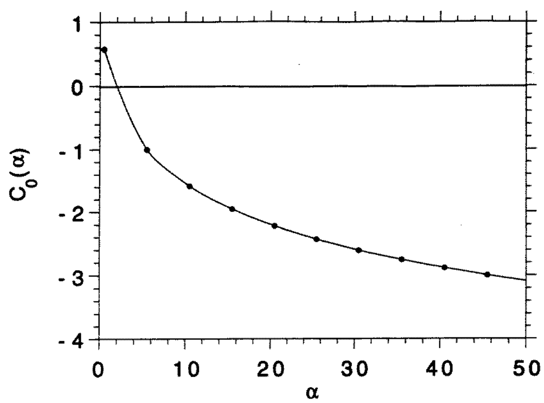

Figure 5. Points: equation (4.13). Line: equation (5.1).

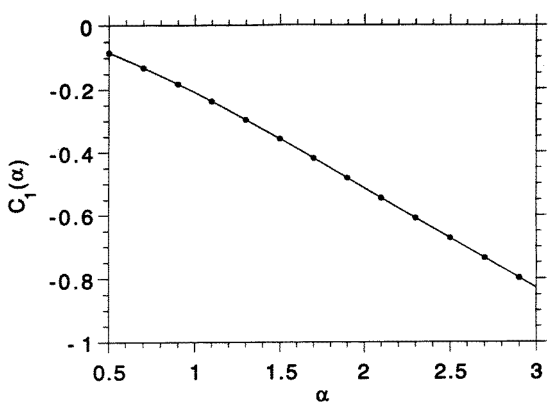

Figure 6. Points: equation (4.16). Line: equation (5.1).

5.3. The points $s = 0$ and $s = -2$

In the Taylor series

$$\zeta_\alpha(-2k + \epsilon) = \zeta_\alpha(-2k) + \epsilon\zeta'_\alpha(-2k) + \mathrm{O}(\epsilon^2) \;\; k = 0, 1, 2, \ldots \tag{5.2}$$

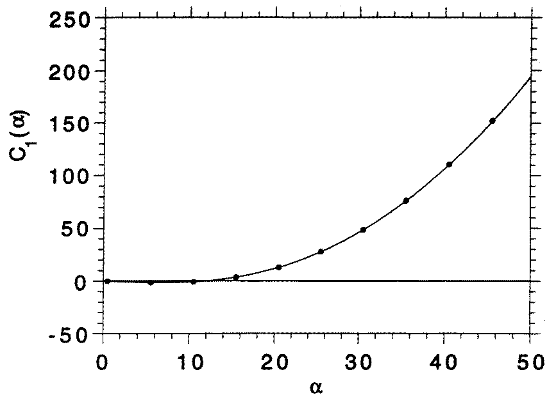

Figure 7. Points: equation (4.16). Line: equation (5.1).

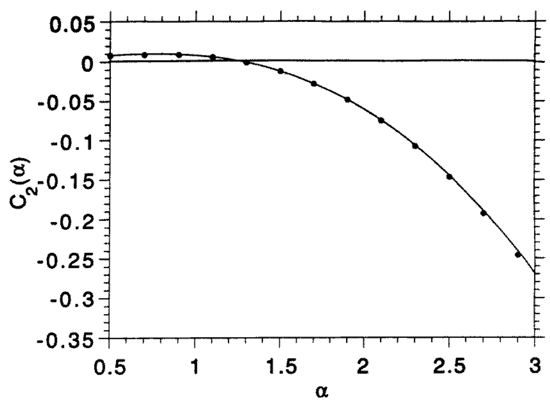

Figure 8. Points: equation (4.20). Line: equation (5.1).

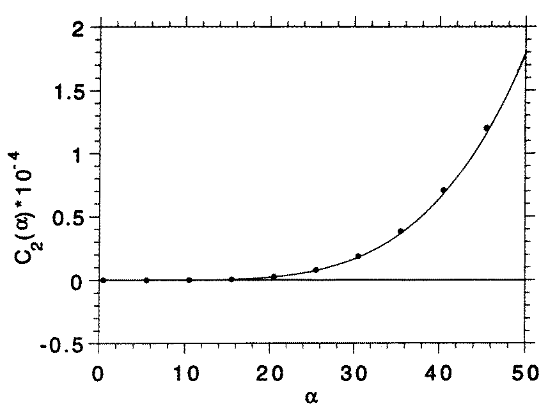

Figure 9. Points: equation (4.20). Line: equation (5.1).

the Hawkins' polynomials $\zeta_\alpha(-2k)$ are known for the first few k values. The derivatives $\zeta'_\alpha(-2k)$ are not known analytically except for $k = 0$. Reproducing the Hawkins' polynomials is a test for our representation (4.5). It is not a test for equation (2.31) which automatically produces the first term in equation (5.2). Here we calculate $\zeta'_\alpha(0)$ and $\zeta_\alpha(-2)$ within our approximation and compare the results with the known formulae for these functions of α.

$s = 0$. Figure 10 plots $\zeta'_\alpha(0)$ versus α for $\frac{1}{2} < \alpha < 3$ calculated from the known result (2.27) and from equation (4.22). Figure 11 does the same thing for the much larger range $\alpha < 50$. We intended to display the sensitivity to $p_{\max}$ by plotting the right-hand side of equation (4.22) for $p_{\max} = 3, 5, 7$ over these same ranges of α. However, these figures show no discernible dependence on $p_{\max}$.

$s = -2$. Figure 12 plots $\zeta_\alpha(-2) = -(\alpha^2 - \frac{1}{4})/4\pi^2$ versus α for $\frac{1}{2} < \alpha < 3$ and does the same with the right-hand side of equation (4.26) which must reproduce this simple function. We use both $p_{\max} = 3$ and 7 in equation (4.26). Note the quite good agreement which improves slightly with increasing $p_{\max}$. Figure 13 does the same thing for $\alpha < 50$ using $p_{\max} = 3, 5, 7$. Over this much greater α range the dependence on $p_{\max}$ becomes far more pronounced. As one would hope, the largest $p_{\max}$ yields the best results.

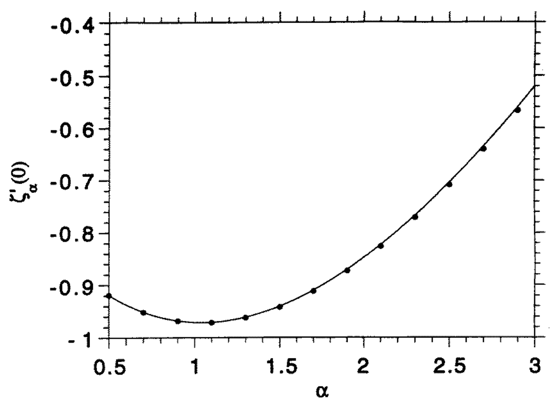

Figure 10. Points: equation (2.27). Line: equation (4.22).

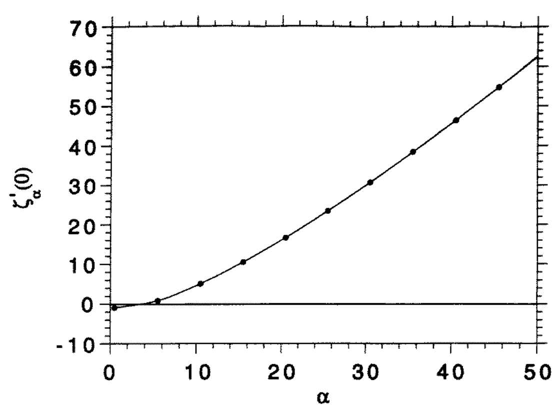

Figure 11. Points: equation (2.27). Line: equation (4.22).

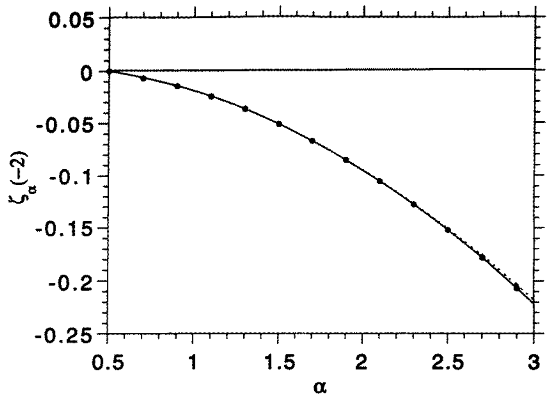

Figure 12. Points: exact value in equation (4.26). Solid line: right-hand side of equation (4.26) with $p_{\max} = 7$. Dotted line: right-hand side of equation (4.26) with $p_{\max} = 3$.

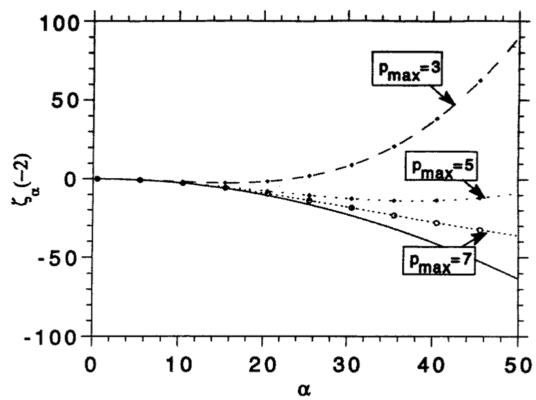

Figure 13. Solid line: exact value in equation (4.26). Circles: right-hand side of equation (4.26) with $p_{\max} = 7$. Diamonds with dots: right-hand side of equation (4.26) with $p_{\max} = 5$. Diamonds with dashes: right-hand side of equation (4.26) with $p_{\max} = 3$.

6. Conclusion

Our primary goal in this paper has been to gain computational control over the Bessel ζ function in the analytic continuation region $\mathrm{Re}\, s < 1$. We reviewed previous results on and analysis of $\zeta_\alpha(s)$ including an integral representation (2.31), and we extended this analysis from the spectrum $\{j_{\alpha n}/\pi\}$ of Bessel function zeros to an arbitrary spectrum of zeros $\{a_n\}$ endowed with a ζ function. We then derived a very different representation (4.5) of $\zeta_\alpha(s)$ by using the McMahon expansion (1.2) of $j_{\alpha n}$. While not a new idea, the approximation to $\zeta_\alpha(s)$ presented and investigated here goes far beyond previous implementations. We explained how this representation is a quasi-power series in the natural expansion parameter $a = (2\alpha - 1)/4$. Some analysis of the first few terms in typical power series was given to observe the power-series machinery in action. We then tested the two representations of $\zeta_\alpha(s)$ numerically, comparing them with one another for selected s (resp. α) with α (resp. s) running over some interval. Also, we compared equation (4.5) with known special values. Overall very good agreement was found. Changes in either representation (i.e. $p_{\max}$ used in equation (4.5) or the N used in equation (2.31)) had a visible but generally small numerical effect, giving a sense of solidity to all of these calculations.

In the course of our study much new information about $\zeta_\alpha(s)$ has been produced. This includes the numerical results in section 5 on $\zeta_\alpha(s)$ itself, results on finite parts of poles, on zeros of $\zeta_\alpha(s)$, and so on. We could just as well have studied other s values and ranges. There is a great deal more work to be done on this very interesting ζ function.

Returning to the physical background of this paper, we remind the reader that the spectrum $\{j_{\alpha n}/\pi\}$ arises in the parallel-wall Casimir problem (1.4) when the Bessel potential $V_{\mathrm{B}}(x) = (\alpha^2 - \frac{1}{4})/x^2$ is used to semiharden one of the walls. A different potential would produce a different spectrum. By changing $V(x)$ continuously one changes the spectrum continuously. Equations (3.5)–(3.8) and indeed the entire analysis in section 3 were designed to exploit this continuity. Section 3 will be useful for future work on boundary semihardening.

Only one-dimensional ζ functions have been considered in this paper. It is quite obvious that higher-dimensional ζ functions can be constructed from $\{j_{\alpha n}/\pi\}$ as well as from other spectra. For example, the one-dimensional Schrödinger equation (1.4) can be replaced by a three-dimensional one

$$[-\Delta + V(\vec{x})]\phi_n(\vec{x}) = w_n^2\phi_n(\vec{x}). \tag{6.1}$$

If $V(\vec{x}) = V_1(x_1) + V_2(x_2) + V_3(x_3)$ then equation (6.1) factorizes into three one-dimensional equations

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}x_i^2} + V_i(x_i)\right]\phi_{in}(x_i) = k_{in}^2\phi_{in}(x_i). \tag{6.2}$$

If $V_i(x_i) = (\alpha_i^2 - \frac{1}{4})/x_i^2$ and we position orthogonal Dirichlet walls at $x_i = L_i$ then the system is a scalar field confined to a rectangular cavity with Bessel walls at $x_{1,2,3} = 0$ and Dirichlet walls at $x_{1,2,3} = L_{1,2,3}$. The momenta in equation (6.2) are $k_{in} = j_{\alpha_i n}/L_i$ and the spectrum in equation (6.1) is

$$w_n^2 = (j_{\alpha_1 n_1}/L_1)^2 + (j_{\alpha_2 n_2}/L_2)^2 + (j_{\alpha_3 n_3}/L_3)^2. \tag{6.3}$$

Obviously one can do this in any number of dimensions. The ζ function constructed from this spectrum is a generalization of the Epstein ζ function [19] in the same way that $\zeta_\alpha(s)$ is a generalization of the Riemann ζ function. We hope to report on such semihard cavities at a later time. For this it will be necessary to bring ζ functions constructed from spectra like (6.3) under adequate control.

Acknowledgments

The authors thank M Bordag for making them aware of the integral representation (2.31) and J S Dowker for helpful correspondence. One of us (AA) thanks the Institut für Theoretische Physik der Universität Heidelberg for its kind hospitality, and as always Penn State Allentown and RDG for support.

Appendix

The first seven coefficients $b_p(a)$ in the McMahon expansion (1.2) of $j_{\alpha n}/\pi$ are given in [9] as polynomials in α. We prefer to list them here as polynomials in the perturbation parameter $a = (2\alpha - 1)/4$:

$$\begin{aligned}
&\pi^2 b_1(a) = a[1 + 2a]\\
&6\pi^4 b_2(a) = a[-3 + a + 28a^2 + 28a^3]\\
&30\pi^6 b_3(a) = a[45 - 12a - 325a^2 + 90a^3 + 996a^4 + 664a^5]\\
&840\pi^8 b_4(a) = a[-9450 + 1395a + 64\,554a^2 - 11\,627a^3 - 143\,920a^4 + 33\,768a^5\\
&\qquad + 222\,368a^6 + 111\,184a^7]\\
&2520\pi^{10} b_5(a) = a[396\,900 - 32\,760a - 2\,665\,713a^2 + 262\,106a^3 + 5\,557\,993a^4\\
&\qquad - 769\,398a^5 - 5\,986\,232a^6 + 1\,218\,704a^7 + 5\,615\,760a^8\\
&\qquad + 2\,246\,304a^9]\\
&55\,440\pi^{12} b_6(a)/a = -196\,465\,500 + 9\,809\,100a + 1\,309\,701\,195a^2\\
&\qquad - 74\,543\,403a^3 - 2\,671\,097\,275a^4 + 200\,770\,955a^5\\
&\qquad + 2\,676\,828\,308a^6 - 303\,476\,228a^7 - 1\,697\,033\,360a^8\\
&\qquad + 305\,494\,288a^9 + 1\,073\,790\,144a^{10} + 357\,930\,048a^{11}\\
&10\,810\,800\pi^{14} b_7(a)/a = 1\,264\,255\,492\,500 - 41\,566\,486\,500a - 8\,393\,930\,663\,625a^2\\
&\qquad + 302\,622\,856\,380a^3 + 16\,937\,841\,355\,578a^4 - 750\,168\,167\,034a^5\\
&\qquad - 16\,565\,084\,652\,545a^6 + 1\,017\,479\,819\,642a^7 + 9\,743\,991\,228\,972a^8\\
&\qquad - 939\,840\,891\,704a^9 - 4\,069\,031\,872\,048a^{10} + 657\,276\,952\,800a^{11}\\
&\qquad + 1\,858\,726\,993\,984a^{12} + 53\,106\,485\,5424a^{13}.
\end{aligned} \tag{A.1}$$

In the notation $b_p(a) = a\hat{b}_p(a)$ we list the constants used in section 4:

$$\begin{array}{ccc}
p & \hat{b}_p(0) & \hat{b}'_p(0)\\
1 & \dfrac{1}{\pi^2} & \dfrac{2}{\pi^2}\\
2 & -\dfrac{1}{2\pi^4} & \dfrac{1}{6\pi^4}\\
3 & \dfrac{3}{2\pi^6} & -\dfrac{2}{5\pi^6}\\
4 & -\dfrac{45}{4\pi^8} & \dfrac{93}{56\pi^8}\\
5 & \dfrac{315}{2\pi^{10}} & -\dfrac{13}{\pi^{10}}\\
6 & -\dfrac{14\,175}{4\pi^{12}} & \dfrac{7785}{44\pi^{12}}\\
7 & \dfrac{467\,775}{4\pi^{14}} & -\dfrac{199\,935}{52\pi^{14}}
\end{array} \tag{A.2}$$

The pole residues $R_n(a)$ of $\zeta_\alpha(s)$ in equation (1.3) for $n \leqslant 7$ are $R_0 = 1$ and (see [1–4, 11] for $n \leqslant 4$)

$$
\begin{aligned}
2\pi^2 R_1 &= -(\alpha^2 - \tfrac{1}{4}) \\
8\pi^4 R_2 &= -(\alpha^2 - \tfrac{1}{4})(\alpha^2 - \tfrac{25}{4}) \\
16\pi^6 R_3 &= -(\alpha^2 - \tfrac{1}{4})[\alpha^4 - \tfrac{57}{2}\alpha^2 + \tfrac{1073}{16}] \\
128\pi^8 R_4 &= -5(\alpha^2 - \tfrac{1}{4})[\alpha^6 - \tfrac{307}{4}\alpha^4 + \tfrac{54\,703}{80}\alpha^2 - \tfrac{375\,733}{320}] \\
256\pi^{10} R_5 &= -7(\alpha^2 - \tfrac{1}{4})(\alpha^2 - \tfrac{25}{4})[\alpha^6 - \tfrac{619}{4}\alpha^4 + \tfrac{284\,357}{112}\alpha^2 - \tfrac{2\,215\,391}{448}] \\
1024\pi^{12} R_6 &= -21(\alpha^2 - \tfrac{1}{4})[\alpha^{10} - \tfrac{1165}{4}\alpha^8 + \tfrac{2\,107\,033}{168}\alpha^6 - \tfrac{38\,525\,995}{224}\alpha^4 \\
&\quad + \tfrac{214\,409\,317}{256}\alpha^2 - \tfrac{3\,530\,432\,987}{3072}] \\
2048\pi^{14} R_7 &= -33(\alpha^2 - \tfrac{1}{4})[\alpha^{12} - \tfrac{955}{2}\alpha^{10} + \tfrac{1\,716\,029}{48}\alpha^8 - \tfrac{492\,664\,765}{528}\alpha^6 + \tfrac{28\,503\,421\,893}{2816}\alpha^4 \\
&\quad - \tfrac{743\,923\,184\,891}{16\,896}\alpha^2 + \tfrac{7\,780\,757\,249\,041}{135\,168}].
\end{aligned}
\tag{A.3}
$$

All R_n for $n \geqslant 1$ vanish at $\alpha = \frac{1}{2}$. All R_n for $n \geqslant 2$ have additional zeros:

$R_2 = 0$ at $\alpha_1 = \frac{5}{2}$

$R_3 = 0$ at $\alpha_{1,2}^2 = \frac{57}{4} \mp 2\sqrt{34}$ or $\alpha_1 = 1.609,\ \alpha_2 = 5.090$

$R_4 = 0$ at $\alpha_1 = 1.512,\ \alpha_2 = 2.7732,\ \alpha_3 = 8.1715$

$R_5 = 0$ at $\alpha_1 = 1.5008,\ \alpha_2 = \frac{5}{2},\ \alpha_3 = 4.0119,\ \alpha_4 = 11.6791$

$R_6 = 0$ at $\alpha_1 = 1.5000,\ \alpha_2 = 2.49\,004,\ \alpha_3 = 3.4531,\ \alpha_4 = 5.3388,\ \alpha_5 = 15.5683$

$R_7 = 0$ at $\alpha_1 = 2.2500,\ \alpha_2 = 6.24\,495,\ \alpha_3 = 11.9816,\ \alpha_4 = 19.0794,$
$\alpha_5 = 45.6863,\ \alpha_6 = 392.25\,777.$

The Hawkins' polynomials $\zeta_\alpha(-2n)$ for $n \leqslant 6$ are (see [1–4, 11] for $n \leqslant 4$)

$$
\begin{aligned}
2\zeta_\alpha(0) &= -(\alpha + \tfrac{1}{2}) \\
4\pi^2\zeta_\alpha(-2) &= -(\alpha^2 - \tfrac{1}{4}) \\
4\pi^4\zeta_\alpha(-4) &= -(\alpha^2 - \tfrac{1}{4})[\alpha^2 - \tfrac{13}{4}] \\
4\pi^6\zeta_\alpha(-6) &= -(\alpha^2 - \tfrac{1}{4})[\alpha^4 - \tfrac{53}{4}\alpha^2 + \tfrac{103}{4}] \\
4\pi^8\zeta_\alpha(-8) &= -(\alpha^2 - \tfrac{1}{4})[\alpha^6 - \tfrac{135}{4}\alpha^4 + \tfrac{3771}{16}\alpha^2 - \tfrac{23\,797}{64}] \\
4\pi^{10}\zeta_\alpha(-10) &= -(\alpha^2 - \tfrac{1}{4})[\alpha^8 - \tfrac{137}{2}\alpha^6 + \tfrac{9063}{8}\alpha^4 - \tfrac{191\,957}{32}\alpha^2 + \tfrac{2\,180\,461}{256}] \\
4\pi^{12}\zeta_\alpha(-12) &= -(\alpha^2 - \tfrac{1}{4})[\alpha^{10} - \tfrac{485}{4}\alpha^8 + \tfrac{12\,425}{32}\alpha^6 - \tfrac{2\,979\,523}{64}\alpha^4 \\
&\quad + \tfrac{27\,234\,823}{128}\alpha^2 - \tfrac{72\,763\,141}{256}]
\end{aligned}
\tag{A.4}
$$

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